

Exact solutions and their interpretation

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Abstract

This is the account of the workshop *Exact solutions and their interpretation* at the 16th International Conference on General Relativity and Gravitation held in Durban, July 15-21, 2001. Work reported in 32 oral contributions spanned a wide variety of topics, ranging from exact radiative spacetimes to cosmological solutions. Two invited review talks, on the role of exact solutions in string theory and in cosmology, are also described.

1 Introduction

In accord with past tradition, the workshop A.1 has once again received the highest number of papers (73) in GR16, thus providing ample proof that the interest in discovering and understanding solutions to the Einstein equations did not fade away with the turn of the millennium. In comparison, a closely related workshop on mathematical studies of the field equations received 48, three workshops devoted to quantum issues totalled 52, mathematical cosmology received 38 and relativistic astrophysics 31, etc. Obviously, quantity does not necessarily mean quality. Also, not all the contributors could have been expected to come to Durban. Despite this, I asked the organizers to allot extra 3 hours to the A.1 workshop, with the welcome side effect that no need of a poster session arose. During 9 hours, 32 original contributions (15 minutes each) were presented. In addition, I turned to Roberto Emparan and Malcolm MacCallum to give reviews (30 minutes each) on exact solutions in string theories and on cosmological models. Their reviews also attracted a great number of GR16 participants, who normally would not have attended talks at the A.1 workshop.

In the Introductory remarks, I very briefly recalled the role of exact – sufficiently explicit – solutions for the development and understanding of general relativity. Despite significant recent advances in demonstrating the *existence* of some quite general classes of solutions which, of course, are “no less exact”, we still rightly welcome special *explicit* formulae. Such questions as location and character of singularities and the global properties of a corresponding spacetime, are best addressed if an explicit solution

is available. A special explicit solution frequently stimulates questions relevant to more general situations. For example, the presence of the Cauchy horizon in the Reissner-Nordström solution inspired a great amount of work on the instabilities of the Cauchy horizons of other types.

The role of the most important exact vacuum solutions such as black-hole solutions, stationary axisymmetric fields, and several classes of exact radiative spacetimes and vacuum cosmological models, in the development of general relativity and relativistic astrophysics, was reviewed recently [1].

In the following, the contributions are arranged in essentially the same order as they were given during the workshop. Although the topics covered were quite diverse, wherever possible, the contributions were grouped together: on exact radiative spacetimes, on the mathematical techniques used in analyzing exact solutions, on cosmological solutions, on the black-hole and other stationary solutions, and three contributions are covered by “Miscellanea”. The two invited reviews are summarized at the end of the article.

With minor exceptions the following text is the combination of the material written by those who gave the talks (with their names in italics), which I have edited, modified in a number of cases, and in all cases had to shorten because of length constraints. I thank all the authors for supplying me with this material and, in particular, for adding the references which make the report more useful. I also thank T. Ledvinka and M. Varadarajan for help.

2 Exact radiative spacetimes

Standing gravitational waves (*H. Stephani*)

H. Stephani proposed an intuitive definition of standing gravitational waves: (i) the constitutive parts of the metric functions should depend on the timelike coordinate only through a periodic factor, and they should also depend on spacelike coordinates (“product structure”); (ii) the time average of some of the metric functions should vanish; in particular, the analogue of the Poynting vector (if there is any) should be divergence-free and the time average of its spatial components should be zero. He then searched for standing gravitational waves among the main classes of exact vacuum solutions – in most cases with a negative result. There are none among the plane waves and the algebraically special solutions. He can find them only as simple subcases of the (generalized) Einstein-Rosen waves. But in particular for the Gowdy universes, the criterion “product structure” gives contradicting results, depending on the choice of the coordinate system. The criterion “Poynting vector” again turns out to be ambiguous. The most probable choice is that connected with the so-called “C-energy” since the energy-momentum pseudotensor gives unconvincing answers, and for the other forms based on a Lagrangian the timelike component of the Poynting vector turns out to be negative. Although Stephani could find Einstein-Rosen waves which meet his two conditions, the coordinate dependence of the criteria remains disturbing. One may expect more standing wave solutions in the classes with only one or none spacelike Killing vector.

Cylindrical spacetimes endowed with angular momentum

(*G. A. Mena Marugán*)

In this work the general solutions to the vacuum Einstein equations with cylindrical symmetry were studied with the symmetry axis being allowed to be singular – it could be endowed with spin and a linear energy density. First a gauge-fixing procedure adapted to cylindrical symmetry that removes all the constraints of canonical gravity was given. The corresponding reduced system has two field-like degrees of freedom which depend only on the radial and time coordinate. The gauge-fixed metric is expressed in terms of these fields. The solution contains also two constant parameters that describe the deficit angle outside the axis and the density of the angular momentum in the axis direction. The symmetry axis is regular if and only if the two constants vanish. It is proved that the gauge-fixed metric is rigorously defined and there exists a Hamiltonian that generates the reduced evolution if one imposes suitable boundary conditions at spatial infinity and on the axis on the two metric fields and, in addition, introduces a certain restriction on the initial data if the angular momentum does not vanish. The explicit form of the reduced Hamiltonian which provides the energy density in the axis direction is also found. This density is preserved in the evolution and turns out to be bounded both from below and above. The considered spacetimes describe cylindrical gravitational waves surrounding a spinning string. Moreover, the deficit angle detected at spatial infinity is equal or greater than the deficit seen close to the axis. Gravitational waves have thus a positive contribution to the energy density. For details, see Ref. [[2]].

On Killing vectors and Bondi expansions (*J. A. Valiente Kroon*)

Asymptotically flat radiative spacetimes have frequently been studied by solving a characteristic initial value problem by means of expansions in inverse powers of a radial parameter (Bondi expansions). If one assumes that the spacetime has a Killing vector field, then by performing similar expansions on the Killing equations it is possible to deduce the asymptotic form of the Killing vectors compatible with asymptotic flatness.[3] Furthermore, one can deduce so-called “constraint equations” for the news function, the mass aspect and the Newman-Penrose constants. The particular form of these quantities is thus restrained. By resorting to the analysis of the orbits of the Killing vectors at null infinity, it is possible to give the solution to the constraint equations. For Killing fields that are non-supertranslational the characteristics of the constraint equations are the orbits of the restriction of the Killing field to null infinity. The possible classes of Killing fields are discussed by analysing their orbits at null infinity. If axial symmetry is imposed, and the spacetime is assumed to be radiative, then one recovers known results on boost-rotation symmetry as the only other admissible symmetry (see Ref. [[4]] and references therein). As an application, the formalism was used to study the Einstein-Maxwell boost-rotation symmetric Petrov type D spacetimes.[5] Valiente Kroon also studied the late time behaviour of the polarisation states, of the Newman-Penrose constants and the Bondi mass. In particular, one finds that the leading term behaviour of the Bondi mass loss is

proportional to the square of the norm of the only non-vanishing Newman-Penrose constant. Since it is of quadrupolar nature, this result resembles, at least formally, Einstein's quadrupole formula.

Boost-rotation symmetric radiative spacetimes and fields: post-GR15 developments (*J. Bičák, P. Krtouš, V. Pravda and A. Pravdová*)

These spacetimes, representing uniformly accelerated objects (e.g. black holes), are significant since in rotationally symmetric, locally asymptotically flat at null infinity electrovacuum spacetimes the only additional allowable symmetry that does not exclude radiation is the boost symmetry (for latest formulations and proofs, see Refs. [[4, 3]] and references therein.) These solutions are the only explicitly known spacetimes in which strong initial data can be chosen on a hyperboloidal initial hypersurface which lead to complete, asymptotically radiative spacetimes in future. Their properties and their use as test-beds in approximation methods, numerical relativity and quantum gravity models have been reviewed recently.[1, 6]

The spinning C-metric is the only known example of such spacetime with Killing vectors not hypersurface orthogonal. It was brought into the canonical form of the boost-rotation symmetric spacetimes, and its interpretation as a metric representing uniformly accelerated, rotating black holes was analyzed.[7] The geodesics in the standard C-metric were also studied.[8]

Most recently, scalar and electromagnetic fields produced by uniformly accelerated (test) charges in de Sitter space were analyzed.[9] New effects arise because de Sitter space has spacelike conformal infinities. The radiative properties of the fields depend on the way in which a given point of \mathcal{I} is approached. With particle horizons, purely retarded fields become singular or even cannot be constructed at the "creation light cone" of the "point" at which a source "enters" the universe. Smooth fields involving both retarded and advanced effects were constructed. In particular, the classical Born's solution for the field of two uniformly accelerated charges in Minkowski space were generalized to the de Sitter universe.[9]

Geodesics in expanding impulsive gravitational waves (*J. Podolský and R. Steinbauer*)

Introduced in a classical paper of Penrose by using his "scissors and paste" method, the explicit solution for spherical impulsive gravitational waves in continuous coordinates was later on given by Penrose and Nutku, and Hogan. It was only recently related explicitly to the impulsive limit of the Robinson-Trautman type N solutions.[10] The latter however has to be considered as formal since the metric tensor contains terms proportional to the square of δ -function. The transformation relating the distributional coordinate system to the continuous one is discontinuous. It is analogous to the one relating the distributional to the continuous form of the metric tensor for impulsive pp-waves which has been analyzed rigorously.[11]

The authors investigated the geodesics in spacetimes with spherical impulsive waves. In particular, employing the continuous form of the metric they found a large class of privileged and simple geodesics which can be related to the geodesics in the distributional form of the metric

“in front” and “behind” the spherical impulse. This provides foundations for a rigorous (distributional) treatment of impulsive Robinson-Trautman solutions of type N .

3 Mathematical methods

Killing vectors, homothetic vectors, and conformal Killing vectors in the Newman-Penrose formalism (*G. Ludwig and B. Edgar*) Finding Killing vectors (KV) or a homothetic vector (HV) of a given metric can be a formidable task. The authors present a new and simpler approach within the Newman-Penrose formalism. Leaning on results [12, 13] previously derived in the Geroch-Held-Penrose formalism they show that the Killing or homothetic equations can be replaced by the NP-Lie equations, a set of equations involving the commutators of the Lie derivative with the four NP differential operators applied to the four coordinates. Provided that these operators refer to a preferred tetrad relative to the KV or HV, the equations can be solved for the Lie derivative of the coordinates, i.e., for the components of the KV or HV. If part of the tetrad (null directions and gauge) can be defined intrinsically, then that part is generally preferred relative to any KV or HV or conformal Killing vector. In case when part of the tetrad cannot be defined intrinsically, the tetrad used is defined only up to one null rotation parameter and/or a gauge factor, and the NP-Lie equations become more involved. The general method remains the same and is still more efficient than conventional methods. See Ref. [[14]] for details. The following contribution is devoted to a closely related problem.

Tetrads and symmetries (*B. Edgar and G. Ludwig*)

Two of the main tools in the search for exact solutions of Einstein’s equations are tetrad formalisms and use of symmetry. The authors propose a method which analyzes efficiently the Killing vector structure of metrics as they are calculated in tetrad formalisms; this method builds on the following results: [15] a vector field ξ is a Killing vector field iff there exists a tetrad \hat{Z}_m^μ ($m = 1, 2, 3, 4$), which is Lie derived with respect to this field, $\mathcal{L}_\xi(\hat{Z}_m^\mu) = 0$. If we are given, in some Z_m^μ , a spacetime containing at least one KV ξ , there exists a tetrad \hat{Z}_m^μ which is Lie derived by that KV and which is given by a transformation linking \hat{Z}_m^μ to Z_m^μ for some values of the six Lorentz parameters. A direct procedure to find KVs is to substitute Z_m^μ via \hat{Z}_m^μ and six arbitrary parameters into $\mathcal{L}_\xi(\hat{Z}_m^\mu) = 0$, and integrate this system for ξ and for all the parameters simultaneously. The explicit KVs and the parameters giving \hat{Z}_m^μ are then obtained. Rather than integrate $\mathcal{L}_\xi(\hat{Z}_m^\mu) = 0$, it is efficient [12] to deal with the equivalent commutator equations for \mathcal{L}_ξ and $\hat{\nabla}_m = \hat{Z}_m^\mu \nabla_\mu$.

Conformally decomposable spacetimes (*J. Carot and B. O. J. Tupper*)

Here spacetime (M, g) is studied whose metric is conformally related to the metric of a 2+2 locally decomposable spacetime (M, \tilde{g}) ; i.e., it is a product of two 2-dimensional spaces. Starting from the fact that the underlying decomposable spacetime cannot possess proper conformal Killing

vectors (CKV) (unless it is conformally flat),[16] the authors show that the CKV of (M, g) must be the Killing and homothetic vector fields of (M, \tilde{g}) and carry out a complete classification of all the different possibilities for the conformal Lie algebra of (M, g) deriving canonical forms for both the metric and the CKV. A number of results on KV and HV fields are reviewed and new ones derived, especially those concerning homotheties and isometries with fixed points. The authors also obtain some classes of physically relevant examples, such as perfect fluids and electromagnetic fields.

Equations for complex-valued, twisting, type- N vacuum solutions, with one or two Killing/homothetic vectors (*D. Finley*)

The goal of understanding solutions of Petrov type N with non-zero twist is still not realized. Although they can hardly represent radiative fields of bounded sources,[17] they attract researches because of a rich mathematical structure. One way to understand them is the use of HH spaces, i.e., complex-valued spacetimes, of Petrov type $N \times N$. They are determined by three PDEs for two functions, λ and a , of three independent variables (and also two gauge functions which may be chosen to be two of the independent variables). As in integrable systems, these form a second order, linear system for λ ; however, here the integrability conditions, involving a , are complicated. Therefore, with the hope of finding new solutions, Finley assumed that these spacetimes admit both one and two homothetic or Killing vectors. The case with one Killing and one homothetic vector reduces equations to two ODEs for two unknown functions of the one remaining variable. Finley described the explicit forms of the metric, tetrad, connections, and curvature for twisting HH spaces of Petrov type $N \times N$, modulo the determining equations. The details can be found in recent Ref. [[18]].

Algebraic approach to the study of spacetimes with an isometry (*C. F. Sopuerta and F. Fayos*)

In this contribution a new formalism for the study of spacetimes possessing a non-null Killing vector ξ is presented. The main quantity in this approach is 2-form $\mathbf{F} = d\xi$ satisfying Maxwell's equations. Considering $(\xi^\alpha, F_{\alpha\beta})$, the equations $\nabla_\alpha \xi_\beta = \frac{1}{2}F_{\alpha\beta}$, $\nabla_{[\alpha} F_{\beta\gamma]} = 0$, $\nabla_\beta F^{\alpha\beta} = J^\alpha = 2R^\alpha{}_\beta \xi^\beta$, and studying the integrability conditions, the authors find that all the components of the Weyl tensor can be expressed in terms of $(g_{\alpha\beta}, \xi^\alpha, F_{\alpha\beta}, T_{\alpha\beta})$, $T_{\alpha\beta}$ being the energy-momentum tensor. An extension of the Newman-Penrose formalism is set up in which a null tetrad adapted to $F_{\alpha\beta}$ is chosen, and the tetrad components of ξ^α and $F_{\alpha\beta}$ are used. The dependence of $C_{\alpha\beta\gamma\delta}$ on $(g_{\alpha\beta}, \xi^\alpha, F_{\alpha\beta}, T_{\alpha\beta})$ is algebraic. Substituting $C_{\alpha\beta\gamma\delta}$ into the second Bianchi identities one obtains first-order equations for the spin coefficients. Therefore, all the compatibility conditions reduce to compatibility conditions for the first-order equations for the spin coefficients. This formalism naturally provides a refinement of the Petrov classification: apart from the algebraic structure of $C_{\alpha\beta\gamma\delta}$ one has the structure of $F_{\alpha\beta}$ (see Ref. [[19]]).

Painlevé analysis in general relativity (*R. Halburd*)

This talk illustrated the use of complex-analytic methods as “detectors” of solvable models. The problem of determining the metric for a non-static shear-free spherically symmetric fluid (charged or neutral) reduces to the problem of determining a one-parameter family of solutions to $d^2y/dx^2 = f(x)y^2 + g(x)y^3$, where f and g are arbitrary. Most of the solutions are special cases of the family given by Sussman. It was shown[20] that Sussman’s solutions correspond to special cases such that the general solution has a simple singularity structure in the complex domain (the Painlevé property). New solutions were also given in terms of Airy functions and Painlevé transcendents. Halburd described all choices for f and g for which the equation above admits a one-parameter family of solutions such that all movable singularities are poles. This is weaker than the Painlevé property and yields a number of solutions described by Riccati equations besides those given in Ref. [[20]]. Conditions for matching these to an external metric were determined.

4 Cosmological Solutions

Hypersurface homogeneous rotating dust models (*A. Krasinski*)

For matter that moves geodesically and with nonzero rotation, Plebański coordinates (P.c.) (t, x, y, z) can be chosen so that the velocity field, the metric and the rotation vector field have the forms $u^\alpha = \delta_0^\alpha$, $g_{0\alpha} = \delta_\alpha^0 + y\delta_\alpha^1$, $w^\alpha = n\delta_3^\alpha$, where n is the number density of the particles of the matter. If any symmetries exist, then every Killing field, expressed in P.c., must have the form $k^\alpha = (C - \phi - y\phi_{,y})\delta_0^\alpha + \phi_{,y}\delta_1^\alpha - \phi_{,x}\delta_2^\alpha + \lambda\delta_3^\alpha$, where $\phi(x, y)$ and $\lambda(x, y)$ are arbitrary functions and C constant. If $\phi = \text{constant}$, then $k^\alpha = Cu^\alpha + \lambda w^\alpha$, and this property is invariant under all the allowed transformations of the P.c. If $\phi \neq \text{constant}$, coordinates can be adapted to this Killing field so that $k^\alpha = \delta_1^\alpha$. When 3 Killing fields exist, a classification of all the metric forms can be provided. The corresponding symmetry algebras can all be assigned to the Bianchi classes; the orbits can be spacelike as well as timelike or null. Apart from a few classes in which the orbit has to be timelike, most of the 25 classes contain a free parameter that measures the tilt of the velocity field with respect to the symmetry orbits, and allows one to consider all possible positions of the orbits relative to velocity, including the nonrotating, isotropic and spatially homogeneous Friedmann limit. The results have all been published by A. Krasinski in J. Math. Physics (see Ref. [[21]] and citations therein).

Spherical inhomogeneous cosmologies with pressure (*J. R. Gair*)

A generalization to the Lemaitre-Tolman-Bondi solution may be obtained describing a spherically symmetric universe filled with dust which has angular momentum. The dust is labeled by a comoving radial coordinate, R , such that each dust particle remains on a surface $R = \text{constant}$. The dust particles within each of these shells move in every tangential direction with a certain angular momentum about the symmetry centre. This angular momentum is conserved and so it is a function of shell label only. It produces an effective tangential pressure, the radial pressure vanishes.

Einstein derived a static solution of this form, Datta and Bondi generalized to the dynamic case, but did not obtain the metric, Magli derived the metric in mass-area coordinates. Now Gair has obtained a new solution[22] in terms of more physical coordinates - the shell label, R , and the proper time, τ , experienced by the dust particles. The equation of motion for the areal radius, r , is of the form $(\partial r / \partial \tau) = \sqrt{E(R) - V(r, R)}$. The potential $V(r, R)$ for a given shell is the potential of a test particle moving in a Schwarzschild-de Sitter metric. The solution contains four arbitrary functions of R - the angular momentum $L(R)$, the mass $M(R)$, the 'energy' $E(R)$ and the initial position of the shells, $\tau_0(R)$. The shape of the potential permits seven different types of evolution for each shell: expansion and recollapse, collapse and bounce, unbounded expansion/collapse, oscillations, static, coasting expansion/collapse and hesitation. Self-similar solutions may be obtained by choosing $M \propto R$, $L \propto R$ and $E = \text{constant}$, and a solution with a null fluid source by letting $L \rightarrow \infty$.

On some perfect fluid solutions of Stephani (*A. Barnes*)

In 1987 H. Stephani derived several solutions for a geodesic perfect fluid flow with zero pressure (dust). Barnes generalized[23] these solutions with non-zero rotation to include a non-zero cosmological constant Λ . All solutions are of Petrov type D and the magnetic part of the Weyl tensor (relative to the fluid flow) vanishes. In general the solutions admit no Killing vectors. The fluid flow is shearing, twisting and expanding. In the limit when the energy density vanishes the solutions reduce to spacetimes of constant curvature. The solutions can be matched across timelike hypersurface to a de Sitter, anti de Sitter or Minkowski spacetime, depending on Λ .

Locally isotropic spacetimes (*M. A. H. MacCallum and F. C. Mena*)

The authors consider spacetimes which locally (at every point in a neighbourhood) have a non-trivial isotropy group. The possible cases are defined in terms of symmetry of the Riemann tensor and its derivatives, following the method introduced by Cartan and Karlhede (see Ref. [[24]]) which gives a unique local characterization of a spacetime. They review and sharpen previous results for the cases with continuous isotropy groups. For the discrete case they analyze the possible maximal isotropy groups allowed by the algebraic structure of the Riemann tensor. The aim is to prove results analogous to those of Schmidt from 1969, who showed that a perfect fluid spacetime which locally admits the reflections in three perpendicular spatial axes as isotropies must admit a 3-parameter group of isometries. The authors obtain several results of this sort for groups of spatial reflections. For example, they show that for a spacetime with a local discrete isotropy group L generated by reflection of a plane, whose energy-momentum tensor is that of a vacuum or perfect or imperfect fluid, if $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta\gamma\delta;\mu\nu}$ and $R_{\alpha\beta\gamma\delta;\mu\nu\rho}$ are invariant under L , then there is a local group G_2 of isometries acting transitively on the surfaces to which the reflection plane is tangent.

On stationary, spherically symmetric inhomogeneous spacetime with matter content obeying $p = \alpha\rho$ (*S. M. Wagh, D. W. Deshkar*)

and P. S. Muktibodh)

The authors consider the shear-free, spherically symmetric, inhomogeneous spacetimes admitting the separable metric of the form[25]

$$ds^2 = -y^2 A^2 dt^2 + 2(y')^2 R^2 dr^2 + y^2 R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where $y = y(r)$, $A = A(t)$, $R = R(t)$. The coordinates are comoving and the spacetime has non-vanishing energy flux. One nontrivial field equation and the equation of state determine $R(t)$. For example, the imperfect matter may obey $p = \alpha\rho$, $\alpha = \text{constant}$. The radial distribution of ρ is unspecified since the field equations do not determine the metric function $y(r)$ and $\rho \propto 1/y^2$. A timelike Killing vector exists when $\dot{R}/A = \text{constant}$. The stationary solution has an application for an inhomogeneous steady state cosmology. In the absence of a timelike KV, the metric can describe an inhomogeneous big bang cosmology.

5 Stationary systems and black holes

On electromagnetic Thirring problems (M. King and H. Pfister)

The authors consider models consisting of two spherical shells of radii a and $R \geq a$, the first one carrying a charge q but no rest mass, the second one mass M but zero charge (the mass shell fulfils the weak energy conditions). To these shells small angular velocities are applied. The coupled Einstein–Maxwell equations were solved exactly in M/R and q/R , and to first order in the angular velocities. The dragging effects and the induced magnetic fields were calculated.[26] The authors confirm some results of Cohen, and Ehlers and Rindler; in contrast to the latter they argue that the results completely fulfil Machian expectations. The collapse limit of the system was also discussed.

Then only one slowly rotating shell carrying both mass and charge was considered. For M/R and q/R for which the weak energy condition is violated, the dragging constant can become negative (antidragging!). Interesting is the behavior of the gyromagnetic ratio: it is very near to $G = 2$ in most of the parameter space, e.g., $1.92 < G < 2.05$ for all $M/2R > 1$ and q/R that the weak energy condition is fulfilled. The authors argue that the “robustness” of the value $G = 2$ and its “coincidence” with $G \approx 2$ for the simplest rotating, charged particles (electron and muon) hints to a deep connection between general relativity and quantum theory.

Exact relativistic treatment of stationary counterrotating dust disks (J. Frauendiener and C. Klein)

Relativistic dust disks with counter-rotating matter have been discussed since the work of Morgan and Morgan. The rigidly rotating single component dust disk was treated numerically and perturbatively by Bardeen and Wagoner and solved analytically by Neugebauer and Meinel in terms of Korotkin’s finite gap solutions. The authors discuss a class of stationary counter-rotating dust disks which interpolates continuously between the rigidly rotating disk and a static disk. The whole metric is given explicitly in terms of theta functions on a hyperelliptic Riemann surface. The metric functions are analyzed analytically and the hyperelliptic functions

evaluated numerically by using spectral methods. Physically interesting limiting cases as the Newtonian, the static, and the ultrarelativistic limit, where the central redshift diverges, are studied. Explicit expressions for the mass and the angular momentum are given. Following Bičák and Ledvinka who studied counter-rotating disks of infinite extension as sources for the Kerr metric the authors show that the matter in the disk can be interpreted as two streams of pressureless matter which move on geodesics of the inner geometry of the disk. The occurrence of ergospheres and the dragging of the inertial frames is also discussed. See Ref. [[27]] for details and references.

Generalized Kerr-Schild metrics and the gravitational field of a massless particle on the horizon (*H. Balasin*)

Following the work of Aichelburg and Sexl, and 't Hooft and Dray on the gravitational field of a massless particle in Minkowski space, and on the horizon of a Schwarzschild black hole, respectively, Balasin investigated the change in the spacetime structure generated by a massless particle moving along the horizon \mathcal{H} of an arbitrary stationary black hole. The generalized Kerr-Schild class characterized by $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + f k_\alpha k_\beta$, $k^\alpha k^\beta g_{\alpha\beta} = k^\alpha k^\beta \tilde{g}_{\alpha\beta} = 0$, $(k\nabla)k^\alpha = (k\tilde{\nabla})k^\alpha = 0$ gives a framework well adapted to the problem. Identifying the null generators of the horizon with the geodetic null vector field k^a and taking the scalar function f to be concentrated on the horizon, i.e. $f = \delta_{\mathcal{H}} \tilde{f}$, allows one to investigate the situation in terms of properties of supporting null hypersurface (the horizon) and the parent geometry represented by $g_{\alpha\beta}$. Balasin obtained a generalized 't Hooft-Dray equation for the reduced “profile” \tilde{f} , whose coefficients are determined by optical scalars of the congruence forming \mathcal{H} and curvature quantities thereon. See Ref. [[28]] and references therein.

The generalized Wahlquist-Newman solution (*M. Mars*)

The Kerr metric and the Wahlquist metric share the property that their Simon tensor vanishes. Recent work has clarified the geometrical meaning of the vanishing of the Simon tensor in terms of a natural relationship between the Weyl tensor and the covariant derivative of the stationary Killing vector (the so-called Papapetrou form). The Kerr-Newman-de Sitter spacetime satisfies the same relationship.[29] The electromagnetic field and the Papapetrou form are also closely connected in this metric. This leads to the likely existence of a charged generalization of the Wahlquist metric – Wahlquist-Newman metric. This was explicitly found in Ref. [[29]]. It was originally given by García using different hypotheses. This family contains eight essential parameters and includes, besides the Wahlquist and Kerr-Newman-de Sitter metrics, the whole Plebański limit of the rotating C-metric. A weakening of the relationship between the Weyl tensor and the Papapetrou form leads to a larger family containing one arbitrary function of one coordinate. The family is generically of Petrov type *II* and possesses only one Killing vector. These metrics admit a well-defined “infinity”, with topology $\mathbb{R} \times S$, where S is a two-dimensional manifold, which may be compact or not. The metrics are in general not asymptotically flat but have a quite rich asymptotic behaviour.

General vacuum solution for axially symmetric stationary space-time (Z. Ya. Turakulov and N. Dadhich)

The authors have obtained the most general vacuum solution for the axially symmetric stationary metric in which both the Hamilton-Jacobi equation for particle motion and the Klein-Gordon equation are separable. It can be transformed into the Kerr-NUT solution. Like the Kerr solution, the Kerr-NUT solution is thus also unique under these assumptions. It is the most general axially symmetric stationary asymptotically non-flat vacuum spacetime admitting regular horizon. See Ref. [[30]] for details.

On the integrability of the exact and perturbative Einstein's equations for axistationary perfect fluids (M. Bradley, G. Fodor, M. Marklund and Z. Perjés)

In this contribution the integrability of stationary axisymmetric perfect fluid spacetimes was investigated using a tetrad approach with the Riemann tensor, the Ricci rotation coefficients and the tetrad vector components as variables.[31, 32] The system of equations is integrable in the generic case according to the Cauchy-Kowalewski theorem. The equation of state is barotropic if and only if $\omega_1\sigma_2 = \omega_2\sigma_1$ holds, where ω_i is the vorticity and σ_i the shear (with tetrad e_1 and e_2 in the " r, θ "-plane). A physically realistic rotating incompressible perfect fluid must be of Petrov type I . If the equation of state is linear, the Petrov type cannot be D . The authors linearize the equations around the interior Schwarzschild solution. The resulting solution of the linearized equations is known to be a first order approximation of a solution to the full non-linear system because of the integrability of the system. An example of the integration was given.

Plane symmetric analogue of NUT space (M. Nouri-Zonoz and A. R. Tavanfar)

The authors introduce a new definition of spacetime symmetry which is in accordance with the symmetry of its curvature invariants and its gravoelectromagnetic fields. Applying this definition, they investigate exact vacuum solutions corresponding to both static and stationary plane symmetric spacetimes using concepts of the (1+3)-decomposition or threading formalism. Demanding the presence of a plane symmetric gravomagnetic field, they find a family of two parameter (m and l) solutions, every member of which is the plane analogue of NUT space. See Ref. [[33]] for details.

New exact solution to the static balance problem (R. B. Mann and T. Ohta)

The static balance problem of finding equilibrium solutions in which the gravitational attraction of two bodies is balanced by their electromagnetic repulsion was solved by Majumdar and Papapetrou. These solutions are extremal in the sense that the mass of each body is equal to its charge in gravitational units, $e_i = \pm\sqrt{4\pi G}m_i$. No one has yet found equilibrium states in which this condition does not hold for both bodies. Within (1 + 1) dimensional ("lineal") gravity the authors considered the motion of two charged gravitating bodies. They solved for the Hamiltonian as a function of the degrees of freedom of the system, which in this case are the centre-of-inertia momentum and the relative

proper separation of the bodies. Demanding that the centre-of-inertia momentum p had vanishing time derivative they found the equation[34] $\kappa/2 \left(\sqrt{p^2 + m_1^2} - p \right) \left(\sqrt{p^2 + m_2^2} - p \right) - e_1 e_2 = 0$. This condition is the generalization of the Majumdar-Papapetrou condition. It depends on the momentum and can be satisfied for some fixed momentum p_c whilst $e_i \neq \pm \sqrt{4\pi G} m_i$.

Kóta-Perjés metrics and their Kerr-Schild class (*M. Vasúth*)

In this work properties of Kerr-Schild metrics and Kóta-Perjés spacetimes were discussed. Gergely and Perjés have examined under which conditions the Kerr-Schild pencil of a vacuum metric generates another vacuum spacetime. Using parameter η , which governs the shear of the null congruence, one can classify the possible solutions of the problem. When η is arbitrary, the solution is either Kasner metric or another Kóta-Perjés metric depending on the curl of the null congruence. In the original paper there are three Kóta-Perjés metrics. In the above classification only two of them were listed. The author shows that the third metric is a mate of the other Kóta-Perjés metric in the class determined by the same special value, $\sin \eta = \pm 1/\sqrt{2}$, of the parameter. Some global properties of this third metric were analyzed.

De Sitter-Schwarzschild geometry (*I. G. Dymnikova*)

Under spherical symmetry, the weak energy condition, the regularity at the center, the asymptotic flatness and the finiteness of the ADM mass define the family of regular solutions which include the class of metrics asymptotically de Sitter as $r \rightarrow 0$ and asymptotically Schwarzschild as $r \rightarrow \infty$. The source is anisotropic perfect fluid which can be interpreted as a vacuum invariant under boosts in the radial direction.[35] According to the author, it connects the de Sitter vacuum in the origin with the Minkowski vacuum at infinity and corresponds to an extension of $\Lambda g_{\mu\nu}$ to the cosmological tensor $\Lambda_{\mu\nu}$ which allows Λ to become evolving and clustering.[36]

With masses $M > M_{crit}$, the spacetime contains an infinite sequence of vacuum non-singular black and white holes whose singularities are replaced with regular cores asymptotically de Sitter as $r \rightarrow 0$. A white hole core can model initial stages of the universe starting from nonsingular big bang, followed by the Kasner-type expansion. The author suggests that the instability of de Sitter vacuum near the regular surfaces $r = 0$ results in a possibility of multiple quantum birth of baby universes inside.[36] A gravitating particle-like structure without horizons with de Sitter vacuum in the origin was applied to estimate the lower limits on sizes of fundamental particles.

Hairy black holes, horizon mass, and solitons (*D. Sudarsky, A. Ashtekar and A. Corichi*)

In this work colored static black hole solutions to the Einstein-Yang-Mills (EYM) equations and hairy black holes in other theories were considered within the isolated horizons framework.[37] Hairy black holes may be regarded as ‘bound states’ of ordinary black holes without hair and

colored solitons. This model is used to predict the qualitative behavior of the horizon properties of hairy black holes, to provide a physical ‘explanation’ of their instability and to put qualitative constraints on the end point configurations that result from this instability. The model explains qualitative aspects of the behavior of surface gravity of the colored black holes. Predictions for the value of the ADM mass of the soliton configurations found often in association with the hairy black holes in, for example, Einstein-Yang-Mills-Higgs theory can be made. These predictions allow one to evaluate the difference in the values of the mass of these solitons in terms of quantities associated solely with the corresponding black holes and the values of the ADM masses of the various Bartnik-McKinnon solitons. The results have been reported in Ref. [[38]], numerical calculations in Ref. [[39]].

Isolated horizons in 2+1 dimensions

(*J. Wiśniewski, A. Ashtekar and O. Dreyer*)

The framework of isolated horizons[37] is here extended to 2+1 dimensional general relativity with negative cosmological constant. An analog of the orthonormal null tetrad is introduced, where l^μ , n^ν are null, l^μ is tangent to the horizon and m^ρ is spacelike. One can then construct the analog of the Newman-Penrose framework.[40] A (weakly) isolated horizon is a pair $(\Delta, [l])$, where Δ is a null surface and two l^μ ’s are equivalent if they differ by a multiplicative constant. Expansion of Δ vanishes and the intrinsic connection 1-form on Δ is Lie-dragged by l^μ . Equations of motion are imposed on Δ and an energy condition on stress-energy tensor is assumed. The consequences of this definition are: intrinsic metric on Δ is Lie-dragged by l^μ , the 0-th law of black hole mechanics holds, and one can always choose a gauge in which electrostatic potential is constant on Δ . Using Hamiltonian methods as in 3+1, the quasi-local definitions of mass and angular momentum can be introduced. The first law of thermodynamics is equivalent to the requirement that the time evolution vector field t^μ gives rise to a Hamiltonian vector field on the phase-space of isolated horizons. However, in presence of non-zero total charge, the appropriate boundary conditions turn out to be more subtle than in 3+1 dimensions. Energy can only be uniquely defined up to an additive function of the electric charge. The concepts introduced were used to analyze the charged generalization of BTZ solution.

Asymptotic behaviour of the proper length and volume of the Schwarzschild singularity

(*A. Qadir and A. A. Siddiqui*)

It has been proved that the proper length of the Qadir-Wheeler suture model, constructed from two sections of two closed FLRW models joined by a Schwarzschild region, goes to infinity as the singularity is approached, while its proper volume shrinks to zero.[41] Asymptotically, the proper length goes as $K^{1/3}$ and the proper volume as $K^{-2/3}$, K is the mean extrinsic curvature. A similar analysis could provide the asymptotic behaviour of the Schwarzschild black hole near the singularity – with K taken as a time parameter. Trivial foliations, as by constant Kruskal time hypersurfaces, or by hypersurfaces of constant values of the compactified Kruskal time, do show that the proper length approaches zero and the

volume approaches infinity. The relevant K-slicing for the Schwarzschild geometry is not so simple, the difference from the suture model being that it has a boundary which limits the part of the region near the singularity whose proper length is to be measured. The authors prove that, as for the suture model, the length does indeed go as $K^{1/3}$ and the volume as $K^{-2/3}$.

6 Miscellanea

The problem of coordinates in general relativity (*A. Chamorro*)

The author defines quasi-Minkowskian locally inertial coordinates (QMLIC) relative to an observer in free fall that is instantaneously at rest at an event by usual requirements: (i) the world line of the observer (WLO) is described in the QMLIC by zero spatial coordinates, time is observer's proper time; (ii) on the WLO the metric coincides with Minkowski metric and its derivatives vanish; (iii) the coordinate lines of the QMLIC have as unit tangent vectors, an orthonormal tetrad which is parallelly transported along the WLO, one vector being WLO's tangent; (iv) if the tangent vectors have components which are the derivatives of the original coordinates with respect to each of the QMLIC, then, in a neighborhood of the event, their absolute differentials differ from zero by a curvature dependent term. The author shows that in a weak gravitational field there are in general infinitely many different QMLICs all of which reduce to the unique Minkowskian coordinates when the curvature vanishes. Once the curvature term and the tetrad are fixed, so are the corresponding QMLIC.

Coframe teleparallel models of gravity. Exact solutions (*Y. Itin*)

Itin studies the coframe teleparallel theory of gravity with the most general quadratic Lagrangian.[42] The coframe field on a differentiable manifold is a basic dynamical variable. A metric tensor as well as a metric compatible connection are generated by the coframe in a unique manner. For a special choice this theory gives an alternative description of Einsteinian gravity - teleparallel equivalent of GR. The field equations of the theory were studied by a "diagonal" coframe Ansatz.[43] Itin obtained the explicit form of all spherically symmetric static solutions of the "diagonal" type to the field equations for an arbitrary choice of free parameters and proved that the unique asymptotically flat solution with Newtonian limit is the Schwarzschild solution that holds for a subclass of teleparallel models.

Matching preserving the symmetry (*R. Vera*)

The matching of spacetimes is usually implicitly assumed to preserve some of the symmetries of the problem. Some previous work performed matchings by taking into account the orbits of the preserved groups at both sides of the matching hypersurface. In the work of Vera,[44] a general definition for such a kind of matching is given. The matching hypersurface is restricted to be 'tangent' to the orbits of a desired group of isometries admitted at both sides. The definition implies that the matching hypersurface inherits the preserved symmetry and its algebraic type. When

matching is preserving a 2-dimensional group of isometries, two functions constructed from the exterior product of the two Killing forms and the differential of one of the Killing forms at one side, and the analogous two functions at the other side, have to coincide at the matching hypersurface. The orbits of the group generate orthogonal surfaces if and only if these two functions vanish. This is related to the ‘circularity condition’ in stationary, axisymmetric interior problems. This implies the absence of convective motions in fluids without flux. The functions vanish in the vacuum exteriors and, therefore, have to vanish also on the matching hypersurface.

7 Invited reviews

The role of exact solutions in string theory (*R. Emparan*)

There are at least two levels at which one can discuss classical exact solutions within the context of string theory: (i) classical solutions of full string theory, and (ii) classical solutions of string theory at low energies. The latter are approximations to the former, where terms in the equations that could give rise to corrections at curvature scales of the order of the string length are neglected. Solutions in (ii) are typically exact solutions of GR, possibly coupled to a variety of scalar and p-form gauge fields. Only a few among these belong to the more stringent class (i).

The techniques for constructing exact solutions to the full classical string theory are rather different from those familiar in GR. Conformal sigma models provide automatically solutions of this sort (see, e.g., Ref. [[45]]). Besides, supersymmetry is often crucial in establishing the absence of stringy corrections to some solutions of the low energy effective action, such as the extremal Reissner-Nordstrom black holes, pp-waves, extremal p-branes, AdS spacetimes etc.[46] Supersymmetry is also of help in solving the field equations, since it reduces them to first order partial differential equations.

Many of the techniques developed for GR can be used in low energy string theory. String theory provides additional solution-generating techniques, based on the existence of dualities (S and T) which relate solutions to different low energy limits of string/M-theory. Other techniques make use of additional, compactified dimensions: for example, performing boosts along them give rise to electric Kaluza-Klein charges.

Exact gravitating solutions were far from the center stage in string theory until around the late '80s. Since '96-'97 they have come to play a crucial role, following the successful microscopic description of black holes using D-branes,[47] and the formulation of the AdS/CFT correspondence, reviewed in Maldacena's contribution to this conference. Besides this, some old and well-known exact solutions of GR have received new clothes through their use in string theory and related higher-dimensional scenarios. For example, the Taub-NUT solution, whose interpretation in conventional GR is obscure, has undergone a transformation (via Euclidean rotation, and the addition of extra dimensions) into a solution to 11D supergravity, which provides the lift to M-theory of a D6-brane.[48] Another interesting example is the use of the C-metric, to describe black hole pair

creation and exact solutions for black holes on branes.[49] Much work extends the techniques of GR to cover Kaluza-Klein reductions, couplings to higher-dimensional p-form gauge fields etc, but it would be desirable to unravel the structure of the simplest situation – *pure* gravity ($R_{\mu\nu} = 0$) in dimensions $d > 4$. Along this line, some solutions with qualitatively new features, such as higher-dimensional rotating black holes,[50] and the instabilities of black branes,[51] have been discovered. A more systematic approach to the construction and study of such solutions appears is still lacking.

Cosmological solutions: selected themes (*M. A. H. MacCallum*)

In this last talk of the workshop, MacCallum first discussed where exact solutions impinge on the cosmological theory. The "standard model" nowadays is the Friedmann-Lemaître-Robertson-Walker (FLRW) model with $k = 0$, with cosmological constant Λ , and with inflation. Its predictions fit the cosmic microwave background and the large-scale structure, but worries remain: (i) because of the large number of "initial" assumptions, such as what is the inflaton, what is the initial quantum state etc; (ii) there are uncertainties in its fit to galactic structures; (iii) no convincing reasons are available for the value of Λ . The role of exact cosmological solutions is that they capture full non-linearity of the Einstein equations and provide tests of the claims of the standard model. For example, phase-plane arguments show that there are anisotropic solutions staying close to the FLRW models for as long as one wishes.[52] Further, there are models from which it is evident that inflation does not necessarily remove anisotropy. It is not clear how generic these models are, not even what "generic" means. Exact solutions may be useful for observational data analysis. For example, number counts can be explained by the inhomogeneous Lemaître-Tolman-Bondi models without an evolution of the sources equally well as by the standard FLRW models with sources evolving. By using exact cosmological solutions one can study (past and future) asymptotics, gravitational waves in a non-linear regime (solutions with G_2 symmetry groups) and see, within a more general framework, what is special about the FLRW models.

In the second part, MacCallum mentioned some progress made in the last decade or so in the field of exact cosmological models. Most of the results are summarized in Refs. [[52, 53]] where a number of other citations can be found (for the references on vacuum cosmological models, see also Ref. [[1]]). New fluid solutions with a hypersurface homogeneity were found, mainly as fixed points of dynamical systems. Other solutions were obtained by generating mechanisms (in particular those with $p = \rho$), by separation Ansätze, or by use of homotheties. More attention has been paid to solutions with Λ . Methods using dynamical systems techniques, symmetries of Hamiltonians, etc are summarized in Ref. [[52]] (see also the contribution of Ugla to the workshop A.3 in these Proceedings). Current efforts are extending to tilted models and cases where self-similar asymptotics does not work. Interesting open questions arise as to whether these new models can be used to test averaging, or to study structure formation.

Finally, MacCallum mentioned the problem of "local" objects in cos-

mology, in particular black holes in spacetimes which are asymptotically FLRW. A simple example is given by joining the Schwarzschild solution to a Whittaker solution in the Einstein universe.[54]

Most of the papers quoted bellow are also contained on the gr-qc archives. Here we give the archive number only if the journal citation is not available.

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